NC Math 3 – Functions and Inverses

Function is a conceptual category within the NC High School Mathematics Standards and is a vital mathematical concept that is distinguished from “expression”, “equation”, “formula” and other mathematical relations. A relation that is a *function* must be

> *A relation that pairs every element in one set, called the domain, with exactly one element of a second set, called the range.*

Always relating an input element to its only output element is the predictive, pattern-like power of functions that make them so useful within mathematics and beyond.

ALGEBRA AND FUNCTION ACROSS K-8

Algebra and *Function* are not separate domains in K-5, as students begin to express patterns and generalizations with variables and graph ordered pairs. Students use variables to express mathematical relationships, and progress toward utilizing equations in which variables represent quantities that vary in relation to one another.

In grades 6 and 7 pattern generalization continues as students investigate ratios and proportions, and grade 8 is when students are introduced to a definition of function. In 8th grade, students experience how functions describe relationships between quantities using words, variables, and graphs; and particular attention is given to linear relationships.

ALGEBRA AND FUNCTION ACROSS 9-12

The HS Instructional Framework provides clarity and outlines the progression of the study of function families across the NC Math 1-3 courses**.**

NC Math 1: Linear, Quadratic, & Exponential
NC Math 2: Quadratic, Square Root, & Inverse Variation
NC Math 3: Exponential, Logarithmic, Polynomial, Rational, Absolute Value, & Trigonometric

The use of variables to describe functional relationships keeps *Algebra* and *Function* connected throughout grades 9-12. Algebra can be related to Function as a tool for better understanding characteristics of functional relationships, when those relationships are expressed symbolically. 8th grade students focus primarily on linear functions, their graphical representations, and their graphical characteristics highlighted within the form $y = mx+b$.

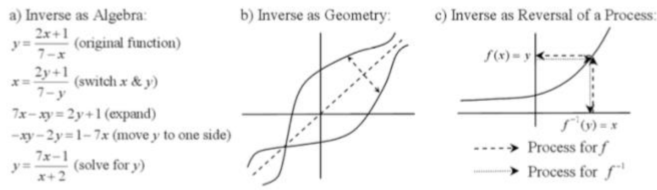
Function notation is not explicit in the standards until NC Math 1. Though the notation is familiar to teachers and ubiquitous across mathematics curricula, “y = f(x)” is a single mathematical sentence that packs a lot of meaning. This sentence shows that “y” and “f(x)” are the same value. It names a function “f”, with a generalized input variable “x”, and a function value or output “f(x)”.

In NC Math 2, students extend their understanding of function to transformations as functions, with two-dimensional domain and range elements within NC.M2.F-IF (think **f(x,y)**, instead of **f(x)**). Here students view a collection of ordered pairs that create an object in the plane, a pre-image, as input values that can be entered into a function to produce the transformed, output image.

NC MATH 3 – UNIT 1, FUNCTIONS AND INVERSES

The NC Math 3 Unit 1 experience of connecting multiple representations of functions (verbal descriptions, numerical values, formulas, and graphs) is a continuation of students’ experiences that begin in 8th grade.

In NC Math 3, students grow their collection of function families from NC Math 2 to include polynomials, rational, absolute values, trigonometric, and logarithmic functions. Also unique to NC Math 3 are piecewise defined functions and the notion of inverse functions (NC.M3.F-BF.4). There are three essential understandings to the concept of *inverse function*. First is the “doing and undoing” nature of the operations that define a function and its inverse that can be determined via algebraic manipulation (e.g. *f(g(x))=x* and *g(f(x))=x*). The second is the graphical reflection of a function's graph over the line y = x to produce the graph of the inverse, so that if *(a,b)* is a point on the graph of function *f*, then *(b,a)* is a point on the graph of f^{-1}. This connects to the third understanding that the domain of a function is the range of its inverse, and vice versa.



Approaches to Engaging with Inverse Functions (Oehrtman et al., 2008)

AN EXAMPLE: THE INVERTIBLE OR NOT? TASK

This task employs the graphical reflection and domain/range relationships of inverses and allows students to use representations of data and graphs to highlight important relationships between functions and their inverse.

t	0	15	30	45	60	75	90	105	120
$f(t)$	0	0.5		1.3	2	2.7		4	
$g(t)$	0	0.5		1.3	2	2.7		4	

a) Complete the table in a way so that f could be invertible and so that g is definitely not invertible
b) Graph both functions and explain from the graph why f is invertible and g is not.
c) Come up with two real life situations that f and g could be representing
d) Find and interpret the value $f^{-1}(4)$ in terms of these contexts

https://www.illustrativemathematics.org/content-standards/tasks/1374

FOCUSING ON A BROADER PERSPECTIVE

A finer notion of *inverse* includes that fact that every function has an *inverse* (operations can be undone and any graph can be reflected about the graph of y=x), but not every function has an *inverse function*, that is an inverse that also satisfies the definition of a function. The Graphs of Compositions task from illustrativemathematics.org allows for students to make this distinction explicit.

It is important to note that in mathematics, calling $\boldsymbol{f}^{-1}$ the *inverse function* of $\boldsymbol{f}$ is equivalent to the relationship of composition in which

$$(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x).$$

This holds value for teachers as they explore and critique curricular materials involving inverse functions, but it isn't a part of the content explored in NC Math 3. Students in NC Math 1, 2, and 3 are not required to study the function operation of composition.

STUDENTS' THINKING ABOUT INVERSE FUNCTIONS

As students progress through the NC Math 1, 2, and 3 courses, they build understanding of different function families and their defining characteristics. In NC Math 3, students engage with the idea that families of functions can be related through the concept of inverse functions. For example, the inverse relationship between quadratic and square root functions or exponential and logarithmic functions. Research has shown that building on students' concpetions of characteristics of different function families (e.g. rate of change, curvature, domain and range) can support their understanding of inverse function relationships (Oehrtman et al., 2008).

There are several places in the NC Math progression where the doing and undoing nature of inverse operations can provide opportunities to build on students' prior knowledge. For example, in NC Math 1 students investigate standard deviation from a conceptual standpoint by learning about the doing and undoing nature around finding the average distance from the mean (squaring and square rooting). In NC Math 2, students explore the relationship between exponents and radical equations (e.g. squaring as an inverse operation to calculating square root). Students are coming into NC Math 3 with a local view of solving for a variable as their context for inverse functions. Taking a more global view of inverse operations, inverse functions require students to connect the algebraic manuevering with other representations which privilege domain and range.

QUESTIONS TO CONSIDER WITH COLLEAGUES

- *What are the big ideas about inverse functions that you want students to have after completing the "Invertible or Not? Task"?*
- *What affordances/constraints do the different representations used in that task offer students when thinking about f, g, f^{-1}, and g^{-1}?*

References

Cooney,T., Beckman,S., & Lloyd,G. (2010). *Developing Essential Understandings of Functions in Grades 9-12*. R. Zbiek (Ed.). Reston, Va.: National Council of Teachers of Mathematics

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics*, MAA Notes (Vol. 73, pp. 27-42). Washington, DC: Mathematical Association of America.

LEARN MORE

Join us as we journey together to support teachers and leaders in implementing mathematics instruction that meets needs of North Carolina students.

NC²ML MATHEMATICS ONLINE

For more information on accessing Canvas learning modules or additional resources please visit http://nc2ml.org/

SUGGESTED CITATION

NC²ML (2018, October). NCM3.1 Functions & Inverses. *Research-Practice Briefs*. North Carolina Collaborative for Mathematics Learning. Greensboro, NC. Retrieved from nc2ml.org/brief-13