NC COLLABORATIVE FOR MATHEMATICS LEARNING

NC^{2}ML PRACTICE BRIEF

REASONING WITH MULTIPLE REPRESENTATIONS

"Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving." (NCTM, 2014, p. 24)

MULTIPLE REPRESENTATIONS

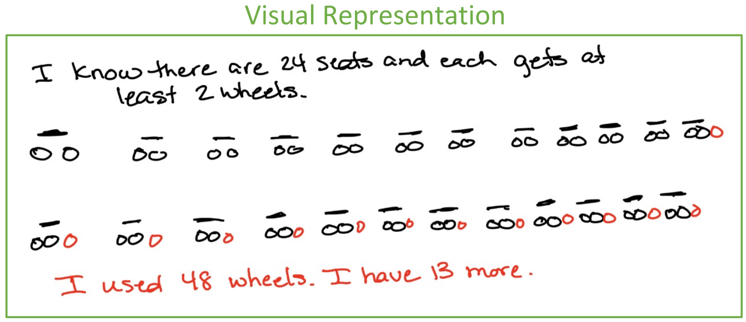
Because of the abstract nature of mathematics, humans rely on representations to describe, explain, and communicate about mathematical ideas. NCTM's Principles to Actions (2014) highlights five different representations that support students' development and understanding of mathematics: visual diagrams, symbolic notation, verbal descriptions, contextual situations, and physical models. Given any mathematical concept, each of these representations reveals different aspects of that concept and can give students access to different information about the concept. When students can reason with a variety of representations to examine mathematics and make connections among those representations, it deepens their conceptual understanding and problem solving capabilities (Lesh, Post, & Behr, 1987).

It is common in the elementary grades for students to explore mathematics content using pictures and hands-on manipulatives as the basis for making sense of more abstract numeric symbols. As students progress through the middle grades and into high school, they should continue to use these visual and physical models and deepen their mathematical understanding of symbolic, numeric, and graphical representations. It is during this time that students connect the concrete to the abstract even more strongly, creating meaning for generalizations such as formulas for area, surface area, and volume (Ferrer et al. 2001; Suh & Moyer, 2007), conceptualizing negative integers in multiple ways (i.e., using two-color counters, a number line, or contexts such as temperature and money (Stephan, 2009)), or making sense of functional relationships from different perspectives.

AN EXAMPLE: THE CYCLE SHOP

You work for a small business that sells bicycles and tricycles. Bicycles have one seat, two pedals, and two wheels. Tricycles have one seat, two pedals, and three wheels. On Monday, there are a total of 24 seats and 61 wheels in the shop. How many bicycles and how many tricycles are in the shop? Show all your work using any method you choose and explain your thinking. (This task comes from Steele and Smith, 2018.)

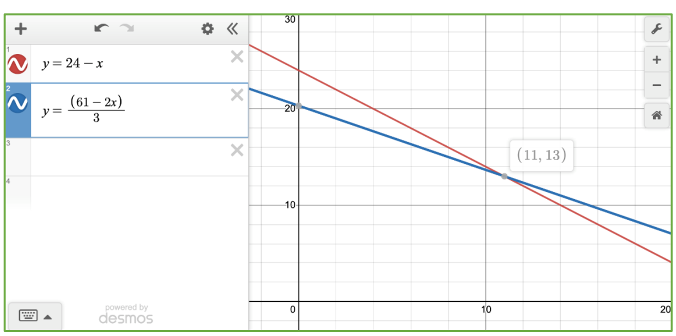
Visual Representation



Numerical Representation

There are 24 bikes + trikes
Start by trying 12 of each
2(12) + 3(12) = 60
so close! Need 61
I'm going to adjust. Go up 1 more trike should be 1 more wheel.
2(11) + 3(13) = 61
11 bikes 13 trikes

Graphical Representation



Symbolic Representation

$$\begin{cases} B + T = 24 \Rightarrow B = 24 - T \\ 2B + 3T = 61 \end{cases}$$

$$2(24-T) + 3T = 61$$
$$48 - 2T + 3T = 61$$
$$48 + T = 61$$
$$\boxed{T = 13 \quad B = 11}$$

RESEARCH ON STUDENT USE OF MULTIPLE REPRESENTATIONS

As students need practice using and making sense of representations (theirs and others), they negotiate meaning for the representations they are using (e.g., Pape and Tchoshanov, 2001). While students' initial representations may lack formal mathematical notation or symbolism, these ideas can be refined and formalized through interactions with peers and the teacher (Webb, Boswinkel, & Dekker, 2008). Additionally, experiences with representing a concept in multiple ways can help support students' abilities to transfer skills and concepts across different contexts (Superfine, Canty, & Marshall, 2009). This is especially important in the later grades when students often tend to have an over reliance on symbolic representations, even when corresponding graphical representations might be more appropriate for thinking and reasoning (Knuth, 2000).

DISCUSS WITH YOUR COLLEAGUES

1. *What do students' representations in the Cycle Shop reveal about their understanding?*
2. *What additional connections can you make across the representations used to solve the Cycle Shop task?*
3. *Think about a task you plan to use in an upcoming lesson and try to solve it using at least four different representations. Reflect on the various ways the representations are connected (i.e., showing the same mathematical ideas in different ways).*

SUPPORTING STUDENT REASONING WITH MULTIPLE REPRESENTATIONS

As teachers of mathematics, we use representations every day. However, we often rely on more abstract and symbolic representations without considering concrete pictorial, visual, or physical representations that are critical for students' development. Additionally, focusing on mathematical procedures limits the nature of the representations that students can develop and use. Incorporating open, cognitively demanding tasks into instruction lends itself to providing opportunities for students to choose ways to represent mathematical situations that are meaningful to them. In addition, it is important that students explicitly make connections within and across multiple representations to support students' deepening understanding of those connections and the ways in which they provide new perspectives to a concept.

Looking across the representations for the Cycle Shop problem, we can see the number of wheels on a bicycle and number of wheels on a tricycle in each. The numerical example shows two wheels and three wheels as one of the factors to determine the total number of wheels (e.g., 2(12) + 3(12) = 60). Students can see and count those wheels in the visual representation and see how they are grouped in the same structure. The structure of the numerical representation is parallel to the structure of the symbolic representation. Finally, looking at the graphical representation students see the ratio of two wheels (bicycle) to three wheels (tricycle) in the slope of the line that represents the number of wheels (and in the form of the equation entered that resulted in that graph). Students can make similar connections across the representations attending to the number of seats and how the solution itself is represented in each. Explicitly discussing these connections – including why they occur, not just that they occur – provides insight to the situation and the mathematics represented within.

When using contextual problems, students should be encouraged to explain situations in their own words. This allows students with varying levels of expertise to access the mathematics and to be seen as competent learners and doers of mathematics. The use of visual and physical representations is critical for emerging bilingual students (Moschovich, 2011) and students with special learning needs (Dieker et al., 2011). The use of such representations allows students to engage with the mathematics and develop meaning on which in-depth conversations can take place. All students can use representations to explain their thinking, make generalizations, and to support claims.

There are many tools today that allow students to create and explore real-time changes in linked representations that are simultaneously visible (e.g., using a slider to change a parameter of a function and examining how it affects the graph, dragging a point in a dot plot and seeing how it changes the mean and median). Exploring linked representations helps students make connections among corresponding features of the various representations (Lapp et al., 2013). (Note: See the Technology brief for examples of dynamically-linked tools.)

References

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